On the dynamical origin of asymptotic t^2 dispersion of a nondiffusive tracer in incompressible laminar flows

Igor Mezić and Stephen Wiggins
Applied Mechanics 104-44, California Institute of Technology, Pasadena, California 91125

(Received 17 December 1993; accepted 24 February 1994)

Using an elementary application of Birkhoff's ergodic theorem, necessary and sufficient conditions are given for the existence of asymptotically t^2 dispersion of a distribution of nondiffusive passive tracer in a class of incompressible laminar flows. Nonergodicity is shown to be the dynamical mechanism giving rise to this behavior.

Consider an incompressible velocity field of the following form:

$$\begin{aligned}\dot{x}&=u(x,y,z,t),\\ \dot{y}&=v(x,y,z,t),\\ \dot{z}&=w(x,y,z,t),\end{aligned}\tag{1}$$

where the velocity field is periodic in one or more of the variables (x,y,z) with the remaining variables (if any) bounded (this setting can be relaxed, as we will explain later). The study of *nondiffusing* passive scalars has gained more attention in the past few years, and this is probably related to the interest in the notion of *chaotic advection* and *stirring* of fluids, which has largely been concerned with the situation of zero molecular diffusion. As examples of recent work along these lines, Jones and Young[1] study dispersion in pipes due to chaotic advection and find t^2 dispersion. Pasmanter[2] and Ridderinkhof and Zimmerman[3] study dispersion in models of shallow tidal flows and find t^2 dispersion. Weiss and Knobloch[4] perform a numerical study of dispersion in modulated traveling waves in binary fluid convection and find a dispersion exponent of 1.93. As their results were numerical, they were only able to compute for a finite length of time. The result in this paper applies in each of these settings and gives conditions under which the dispersion should behave asymptotically (in time) like t^2. As such, it may also prove to be a useful guide for numerical investigations.

We denote the domain of the velocity field generally by A. We assume that the dependence of the velocity field on time is either periodic or quasiperiodic (time *independent* velocity fields are also permitted). In light of the nature of the assumed time dependence we rewrite (1) as

$$\begin{aligned}\dot{x}&=u(x,y,z,\theta),\\ \dot{y}&=v(x,y,z,\theta),\\ \dot{z}&=w(x,y,z,\theta),\\ \dot{\theta}&=\omega,\end{aligned}\tag{2}$$

where θ is an n vector of angular variables ($n=1$ indicates time periodicity) and ω is a constant n vector (the frequencies). Thus the velocity field (2) is defined on the compact set $A\times T^n$, where T^n denotes the n torus. We denote the flow generated by (2) by $\phi_t(x,y,z,\theta)$.

Choose any component of the velocity field in which the corresponding coordinate is periodic (for the purpose of dispersion studies, we will view this coordinate as increasing without bound, i.e. as $\mathbb{R}$). For definiteness, we assume that the z component satisfies this requirement, i.e., the velocity field is periodic in z. We can rewrite the z component of (2) in integral equation form as

$$z(t)-z(0)=\int_0^t w[\phi_\tau(x,y,z,\theta)]d\tau.$$

The mean square displacement or *dispersion* of the z component of (2) of an ensemble of points under the flow is given by

$$\langle[z(t)-z(0)-\langle z(t)-z(0)\rangle]^2\rangle\equiv D_z(t),$$

where the average indicated by the angle brackets is defined as

$$\langle z(t)-z(0)\rangle\equiv\int_{A\times T^n}[z(t)-z(0)]p\ d\mu,$$

$p=p(x,y,z,\theta)$ is the initial distribution of points (assumed to be bounded and integrable on $A\times T^n$), and $d\mu$ denotes the measure or "volume element" on $A\times T^n$. Incompressibility of the flow implies that the flow is "measure preserving," which is important for the application of Birkhoff's ergodic theorem.

We are interested in determining the asymptotic behavior of the dispersion. We have the following calculations:

$$\begin{aligned}\lim_{t\to\infty}\frac{D_z(t)}{t^2}&=\lim_{t\to\infty}\left\langle\left(\frac{1}{t}\int_0^t w[\phi_\tau(x,y,z,\theta)]d\tau-\left\langle\frac{1}{t}\int_0^t w[\phi_\tau(x,y,z,\theta)]d\tau\right\rangle\right)^2\right\rangle,\\ &=\left\langle\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t w[\phi_\tau(x,y,z,\theta)]d\tau-\left\langle\lim_{t\to\infty}\frac{1}{t}\int_0^t w[\phi_\tau(x,y,z,\theta)]d\tau\right\rangle\right)^2\right\rangle,\\ &=\langle[w^*(x,y,z,\theta)-\langle w^*(x,y,z,\theta)\rangle]^2\rangle\equiv a.\end{aligned}$$

The mathematical manipulations in these calculations are justified as follows:

(1) The passage from the first to the second line is justified by the fact that the function w is bounded and integrable on $A \times T^n$.

(2) In the second line, the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t w[\phi_\tau(x,y,z,\theta)]d\tau \equiv w^*(x,y,z,\theta)$$

exists for all points in $A \times T^n$ by Birkhoff's ergodic theorem (see Ref. 5), *with the possible exception of a set of μ-measure zero*. This limit is the time average of the function w along the fluid particle trajectory that starts at the point (x,y,z,θ). Moreover, Birkhoff's ergodic theorem also guarantees that this limit is integrable. This, together with the boundedness of w, implies that the quantity a defined above is finite, and if it is nonzero, we can conclude that the dispersion of the ensemble of particles in the z direction behaves asymptotically like t^2.

The nature of the coefficient a gives some insight into the dynamical mechanism giving rise to t^2 dispersion. It is easy to see that since the expression inside the angle brackets defining a is non-negative, $a=0$ if and only if $w^* = \langle w^* \rangle$ on the support of p, i.e., on the set of points for which $p(x,y,z,\theta)$ is *nonzero*, with the possible exclusion of sets of measure zero. However, $\langle w^* \rangle$ is a constant. Therefore, we can make the following conclusions. Let $C \subset A \times T^n$ denote the support of $p(x,y,z,\theta)$. Then, if $w^*(x,y,z,\theta)$ is not constant almost everywhere on C, $D_z(t) \sim t^2$ as $t \to \infty$. Now assume that the flow is ergodic. Then $\langle w^* \rangle = \langle w \rangle$ almost everywhere. Therefore, $a=0$. So, a necessary condition for t^2 dispersion is the nonergodicity of the flow.

Clearly, this same argument can be repeated for any component of the velocity field corresponding to a coordinate that is periodic. We end this note with some final observations.

(1) Note that our result is independent of the Reynolds number. We are dealing solely with kinematical considerations.

(2) If a particular coordinate direction is bounded (as opposed to periodic) the time average of the corresponding velocity component in that direction is zero. This implies that the dispersion in that direction does *not* behave asymptotically like t^2, and actually, because of boundedness, it cannot behave as any positive power of t.

(3) Compactness of the domain $A \times T^n$ on which the flow is defined, as well as "compactness in time," i.e., quasiperiodicity in time, was important for the application of Birkhoff's ergodic theorem. However, similar conclusions can be drawn for flows that have a volume preserving symmetry (see Ref. 6), such as certain pipe or duct flows. More precisely, consider a pipe flow where the cross-sectional flow decouples from the axial flow as follows:

$$\begin{aligned} \dot{x} &= u(x,y,\theta), \\ \dot{y} &= v(x,y,\theta), \\ \dot{z} &= w(x,y,\theta), \\ \dot{\theta} &= \omega, \end{aligned} \tag{3}$$

where now we allow z to be unbounded (the axial coordinate), but $(x,y) \in A$, where A is a compact subset of $\mathbb{R}^2$ (for an example, see, e.g., Ref. 1 for a flow of this type where t^2 dispersion has been observed). If we let $\phi_t(x,y,\theta)$ denote the flow generated by the $x-y-\theta$ component of (3) then the same arguments as above can be applied to show that the dispersion in the axial direction behaves asymptotically as t^2, provided the initial distribution of points in the cross section is in a region where the time average of the axial velocity is not constant.

More generally, similar results can be stated for unbounded domains, using a version of Birkhoff's ergodic theorem for unbounded domains (see Ref. 7), and requiring that the velocity field be integrable on the domain.

(4) There has been much work done in the last ten years on chaotic fluid particle dynamics in two dimensional, time-periodic velocity fields. In such situations one typically sees a mixture of regular and chaotic regions of fluid particle motions in the flow. Our result implies that if one takes an initial distribution of points that is, roughly speaking, not entirely contained in a *single* regular or chaotic region, then the asymptotic behavior of the dispersion will go like t^2.

(5) The case of a *diffusive* tracer is outside the scope of this note. However, we want to mention a preliminary result of work in progress.[8] This work indicates that our t^2 dispersion result in the nondiffusive case is important for studying the diffusive case in the high Péclet number limit. In particular, for the class of flows considered in this paper, it is possible to show that t^2 dispersion of the convective part of the problem implies a Pe^2 dependence of the effective diffusivity coefficient in the diffusive case. In fact, the dependence of the effective diffusivity on the Péclet number can be determined solely on the knowledge that the convective part of the problem exhibits t^2 dispersion.

Numerical studies of chaotic incompressible flows typically show regions of chaotic and ordered behavior, therefore nonergodicity. In order to study dispersion in chaotic regions, the usual procedure is to consider an initial distribution of points that is entirely contained in what seems to be a chaotic (and supposedly ergodic) region. From the above result, in the diffusive case it would not make any difference whether the points were initially placed in only one ergodic region, or spread out over several ergodic regions. The only importance lies in the fact that in the related nondiffusive problem, when the particles are placed initially in several ergodic regions, the nondiffusive dispersion behaves like t^2 at large times. Then the Pe^2 regime is obtained in the diffusive problem. In this context, we consider our results to be typical for a large class of laminar flows.

ACKNOWLEDGMENTS

We would like to thank John Brady for a critical reading of this note. This work was supported by an NSF Presidential Young Investigator Award, ONR Grant No. N00014-89-J-3023, and AFOSR Grant No. AFOSR910241.

[1] S. W. Jones and W. R. Young, "Shear dispersion and anomalous diffusion by chaotic advection," submitted to J. Fluid Mech. (1990).
[2] R. Pasmanter, "Deterministic diffusion, effective shear and patchiness in

shallow tidal flows," in *Physical Processes in Estuaries*, edited by J. Dronkers and W. Van Leussen (Springer-Verlag, New York, 1988), pp. 42–52.
[3]H. Riddeninkhof and J. T. F. Zimmerman, "Chaotic stirring in a tidal system," Science **258**, 1107 (1992).
[4]J. B. Weiss and E. Knobloch, "Mass transport and mixing by modulated traveling waves," Phys. Rev. A **40**, 2579 (1989).
[5]V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968).
[6]I. Mezić and S. Wiggins, "On the integrability and perturbation of three-dimensional fluid flows with symmetry," J. Nonlinear Sci. **4**, 105 (1994).
[7]U. Krengel, *Ergodic Theorems* (Gruyter, New York, 1985).
[8]I. Mezić, J. F. Brady, and S. Wiggins, "Maximal effective diffusivity for time periodic incompressible fluid flows," Caltech preprint.